UNDERSTANDING ANCIENT MATH THROUGH KEPLER: A PRESENTATION

OF A FEW GEOMETRIC IDEAS FROM *THE HARMONY OF THE WORLD*

Christopher Arthur, B.S.

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APPROVED:

Nick Anghel, Major Professor
Doug Brozovic, Committee Member
John Allen, Committe Member
Neal Brand, Chair of the Department of
Mathematics
C. Neal Tate, Dean of the Robert B. Toulouse
School of Graduate Studies

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Euclid's geometry is well-known for its theorems concerning triangles and circles. Less popular are the contents of the tenth book, in which geometry is a means to study quantity in general. Commensurability and rational quantities are first principles, and from them are derived at least eight species of irrationals. A recently republished work by Johannes Kepler contains examples using polygons to illustrate these species.

In addition, figures having these quantities in their construction form solid shapes (polyhedra) having origins though Platonic philosophy and Archimedean works. Kepler gives two additional polyhedra, and a simple means for constructing the "divine" proportion is given.

Contents

1 Introduction

Of the mathematics of ancient Greece, the geometry of Euclid is well-known for its theorems and postulates concerning triangles and circles. Less well-known and perhaps, less accessible, are the contents of the tenth book, in which geometry is a means to study the science of proportion or quantity in general. Commensurability and the concept of a rational quantity are the first principles, and from them not only comes the concept of an irrational quantity but at least twelve species of irrationals. In a work by Johannes Kepler, recently republished, there are excellent examples using regular inscribed polygons to illustrate the properties of the various Euclidean species of quantity. In addition, the figures having these quantities in their construction exhibit a potential to form solid shapes (polyhedra) having origins through Platonic philosophy and the works of Archimedes. In addition, two additional polyhedra would seem to come from Kepler, and a simple but refreshing means to construct the "divine" proportion is given and has the application of serving to construct the regular pentagon.

2 Euclidio-Keplerian measurement:

Determining to which species a quantity belongs begins with having one known quantity, such as a line segment deemed to be of unit length. Euclid would call this quantity to be rational ("expressible"). By comparison to this one quantity, other quantities have classification. If a quantity is a multiple of the one, or if any two of them share a common divisor, then they are commensurable. Any quantity commensurable with a rational is also rational. Although, by the terminology of Euclid, two rational quantities are not necessarily commensurable; they may merely be commensurable in square. Quantities that have rational squares are also rational ("expressible in square"), by Euclid. If a quantity is neither commensurable nor commensurable in square to a rational, then it is called irrational.

Among the irrationals are the six degrees of apotomes and six degrees of binomials, those of the first and fourth kind arise the most often in the regular and constructible polygons. Kepler outlines eight "degrees of knowledge". The eight degrees are all described in the tenth book, although with different names. A line segment belongs to the first degree of knowledge if it is equal in length to the diameter of a circle. (As Euclid fixes a segment and calls it a rational, so does Kepler fix a diameter of a circle). The second degree contains segments commensurable with the diameter. The third degree contains those having squares that are commensurable with the diameter. The fourth contains segments somehow in proportion with respect to the diameter. The fifth, sixth, seventh, and eighth degrees of knowledge are more subtle.

3 Degrees of knowledge and species of quantity

The following passages describe geometric constructions and settings in which arise the quantities that Kepler and Euclid define. In many cases the definitions parallel one another, but in some cases degrees of knowledge are broad enough to encompass many of the species from Euclid. Almost all the examples involve a diameter taken as a rational quantity with which to compare other quantities.

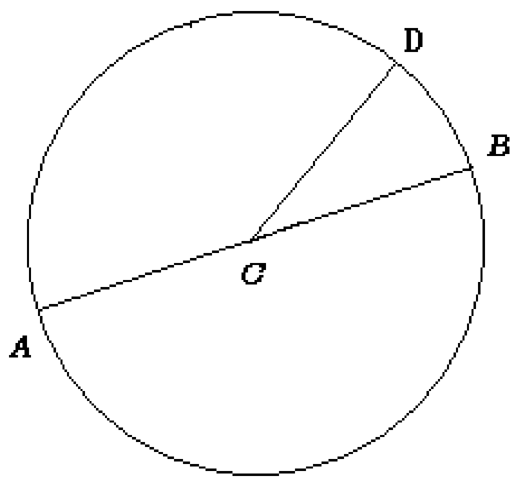
3.1 The first degree of knowledge

Any line segment equal in length to the diameter represents a quantity of the first degree of knowledge. Its magnitude is "expressible" (Kepler) in terms of the diameter and it is also "rational" (Euclid) with respect to the diameter.

Example: Let C be the center of a circle. Let segment AB be a diameter of circle C. Let $|\mathsf{EF}| = |\mathsf{AB}|$. Then since AB is a diameter, EF is of the first degree. QED

3.2 The second degree of knowledge

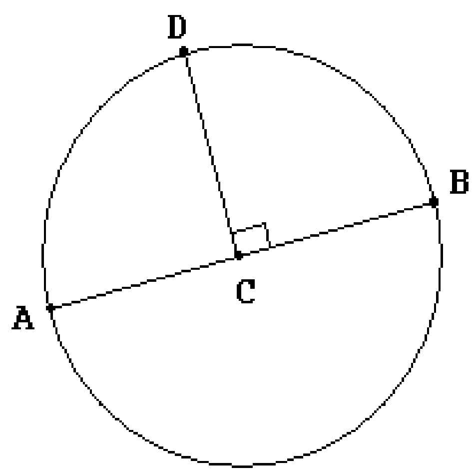
Any line segment not equal to yet commensurable to the diameter is rational and expressible. Alternatively,if for any line segment a second, shorter segment divides both the first segment and the diameter, then the first segment (and the second as well) belong to the second degree of knowledge. The radius of a circle is such a segment.



Example: Let A, B and C be as before. Let D be any other point on the circle C. Then segment CD is a radius, so $|\mathsf{CD}| = \frac{1}{2}|\mathsf{AB}|$. The value $\frac{1}{2}$ is rational, so CD is of the second degree. QED

3.3 The third degree of knowledge

If a line segment is incommensurable to the diameter but its square is commensurable, then the segment is still rational (by Euclid) and is expressible in square (by Kepler). Such segments belong to the third degree. The chord marked by a right, central angle is of the third degree of knowledge. (By Pythagorean theorem)



Example: Let A, B and C be as before. Construct segment DC perpendicular to AB so that DC is a radius. Since $|\mathsf{AB}| = 1, |\mathsf{DC}| = |\mathsf{CB}| = \frac{1}{2}$. By the pythagorean theorem, $|\mathsf{DB}|^2 = |\mathsf{DC}|^2 + |\mathsf{CB}|^2$. Hence, $|\mathsf{DB}| = 2 * \frac{1}{4}$. So $|\mathsf{DB}|$

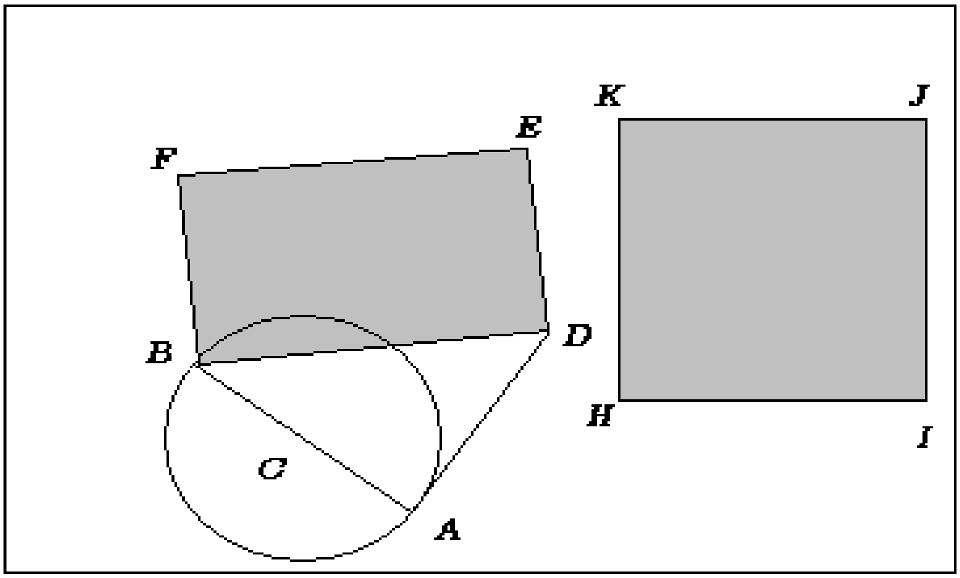
is of the third degree since $\frac{1}{2}$, the value of the square, is rational.

3.4 The fourth degree of knowledge

A single line or a pair of lines can be of the fourth degree. For a single line, its length is of the sort that Euclid calls Medial. A line segment is medial if: 1. it is not of the first three degrees, 2. its square is equal to the product of two other segments that are either one of the third degree and the second, or both of the third degree. However, if both are of the third degree, then the square of their quotient cannot be equal to the quotient of any perfect squares.

The following example describes a medial quantity in relation to two rational quantities, one being expressible and the other being expressible in square:

Consider again a circle of known diameter. The square having side length equal to the diameter has a diagonal which is expressible in square but incommensurable to the diameter. The rectangle having as width the diameter and as length this diagonal is equal in area to a square having a side of medial length. Thus the length of such a side belongs to the fourth degree of knowledge:



Let A, B and C be as before. Let $\mathsf{AD} \perp \mathsf{AB}$ and $|\mathsf{AD}| = |\mathsf{AB}|$. Then BD is of the third degree, by pythagorean theorem. Let BDEF be a rectangle so that $|\mathsf{BF}| = |\mathsf{AB}| = |\mathsf{DE}|$. Suppose HIJK is a square having equal area to BDEF. Then $|\mathsf{HI}|^2 = |\mathsf{BF}| * |\mathsf{DB}|$, which is not a rational quantity since $|\mathsf{DB}|$ is not rational and $|\mathsf{BF}|$ is rational. $|\mathsf{HI}| = \sqrt{|\mathsf{BF}| * |\mathsf{DB}|}$, also not rational, so HI is a medial line since $\frac{|\mathsf{BF}|}{|\mathsf{DB}|} = \sqrt{2}$ (not a ratio of perfect squares).

Also, taken as a pair, the diameter and the diagonal may belong to this degree. Alternatively, Kepler defines a pair of lines to be in this degree if they meet the following criteria:

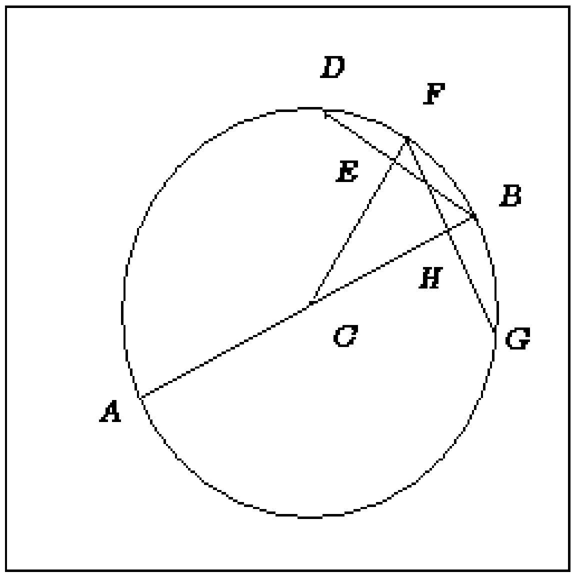
1. Two lines d1 and d2 are incommensurable. 2. d1 + d2 = diameter of the circle (rational). 3. d1 * d2 = c1 * c2 such that c1 and c2 are commensurable.

3.5 The fifth degree of knowledge

A pair of line segments belongs to the fifth degree if:

1. They are neither expressible nor commensurable. 2. The sum of their squares is expressible. 3. Their product is expressible.

For example, consider a dodecahedron inscribed in the circle. This can be accomplished by first constructing the inscribed hexagon then by bisecting the sides of the hexagon. The pair of lines belonging to the fifth degree are the side of the dodecahedron and the leg formed by completing the triangle having the diameter as hypothenuse.



Let A, B and C be as before. Construct chord DB so that $|\mathsf{DB}| = |\mathsf{CB}|$. Let E be the midpoint of DB, and extend CE to form radius CF.

Claim: The pair AF and FB is of the fifth degree.

Proof:

(1) Let FG be the chord intersecting AB at H and having length equal to CB. By symmetry, CB bisects FG and $\mathsf{CB} \perp \mathsf{FG}$.

$|\mathsf{AF}|^2 = |\mathsf{FH}|^2 + |\mathsf{AH}|^2$. Let r be the radius. $|\mathsf{FH}| = \frac{\mathsf{r}}{2}$, by symmetry. $|\mathsf{AH}| = \mathsf{r} + |\mathsf{CH}|$. $|\mathsf{CH}| = \mathsf{r}\frac{\sqrt{3}}{2}$. $|\mathsf{AH}| = \mathsf{r}(1 + \frac{\sqrt{3}}{2})$. So, $|\mathsf{AF}| =$

$\sqrt{\frac{r^2}{4} + r^2(1 + \frac{\sqrt{3}}{2})^2}$. Clearly, AF is neither expressible nor expressible in square since it contains the term $\sqrt[4]{3}$. Similarly, FB can be computed to be $r\sqrt{2 - \sqrt{3}}$. That FB and AF are incommensurable is left to the reader.

(2) Since $\angle AFB$ is a right angle, $|FB|^2 + |FA|^2 = |AB|^2$. Thus by the pythagorean theorem, the sum of the squares is expressible (of the first degree).

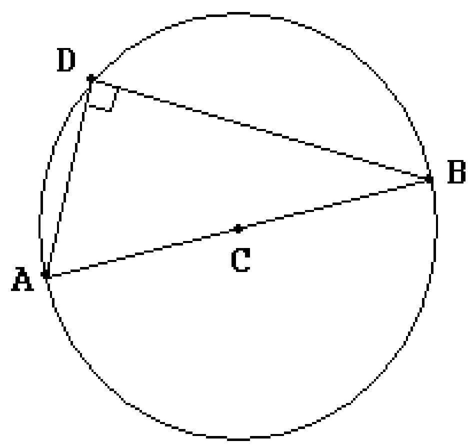
(3) By similar triangles, FB:AB::BE:AF. So, $|FB| * |AF| = |AB| * |BE|$. Since $|DB| = 2|BE|$, $|AB| * |BE|$ is expressible, and thus $|FB| * |AF|$ is expressible.

QED

3.6 The sixth degree of knowledge

This degree classifies a pair of line segments by two quantities derived from their lengths. Like the fifth degree, the two derived quantities are the sum of their squares and their product. If one of these is medial and the other is expressible, then the pair of lines belongs to this degree.

Example: Consider the sides of the right triangle having the diameter as hypothenuse, longer leg of length the square root of seven and shorter leg of length the square root of two. Then the two legs are expressible in square and otherwise incommensurable. The sum of their squares, by the pythagorean theorem, is expressible. Their product, the square root of fourteen, is not expressible, so the square root of the product is not expressible in square and is therefore a medial length. Thus the two legs together are of the six degree.



Let A, B and C be as before, but let $|\mathsf{AB}| = 3$. Suppose that on the circle is placed point D so that $|\mathsf{AD}| = \sqrt{2}$ and $|\mathsf{DB}| = \sqrt{7}$.

Claim: DB and AD together are of the sixth degree.

Proof: By the pythagorean theorem, $|\mathsf{DB}|^2 + |\mathsf{AD}|^2 = |\mathsf{AB}|^2$, so the sum of squares is an expressible quantity. Since $\sqrt{|\mathsf{AD}| * |\mathsf{DB}|} = \sqrt{\sqrt{2}\sqrt{7}} = \sqrt[4]{14}$, the product is not expressible in square and is therefore medial. Thus AD and BD are of the sixth degree. QED

Kepler does not provide any example of this degree, although it does serve to define the well-explored subsequent degrees.

3.7 The seventh degree of knowledge

A pair of lines belongs to the seventh degree of knowledge if they are not commensurable, and both their product and the sum of their squares are medial quantities (belonging to the fourth degree of knowledge).

3.8 The eighth degree of knowledge

Line segments belonging to this degree are not under rigid definitions. Certainly, they do not belong to the previous seven degrees. Generally, they occur as sums and differences among segments of the previous degrees.

4 Finding polyhedra

4.1 Determining the existence of polyhedra:

The task of determining which regular polygons may join together to form polyhedra is greatly simplified by a few observations and requirements. First, at any vertex of a polyhedron, the sum of the angles of the meeting polygons is less than a full circle. Otherwise, the polygons would together not make a solid angle. Second, the sides of all polygons must be of the same size, regardless of the area of any polygon. This ensures that neither gaps nor hanging fringes occur. Third, the set of polygons meeting at any one vertex is the same as that of any other. This serves well to define the Archemedian solids (below). Fourth, odd-sided polygons may not meet with only two other kinds of polygon, as a consequence of the previous point. Fifth, at least three polygons meet to form a solid angle. Only two of them would make a plane angle. With these observations, the infinity of how polygons may meet at a single vertex is greatly reduced to just a relative few possibilities.

4.2 The Platonic solids

A Platonic solid is a polyhedron having faces which are regular polygons all congruent to one another. A cube, for example, has faces that are squares all of equal area, and it is one of the five Platonic solids.

The tetrahedron (four faces), octahedron (eight faces), and the icosahedron (twenty faces) are all solids having faces which are equilateral triangles. The fifth Platonic solid is the regular dodecahedron (twelve faces) having faces which are regular pentagons.

To arrive at this collection, what is helpful is to consider how polygons may meet at a given vertex so to form a solid angle. Three equilateral triangles meet at a tetrahedron vertex. Four of them meet at an octahedron vertex. Five meet at an icosahedron vertex. Six of them do not form a solid angle since together they make a plane surface. Going on to other polygons, three squares meet at a cube vertex, and four squares fill the plane. Three pentagons meet at a dodecahedron vertex, and four of them more than fill the plane. No number of hexagons make a solid angle since three make a plane.

4.3 The Archimedian solids

An Archimedian solid is a polyhedron having both faces which are regular polygons (congruent if having the same number of sides) and vertices which are all congruent. The "truncated cube", for example, has faces which are either regular octagons or equilatral triangles. The octagons are all congruent and the triangles are all congruent. Furthermore, at each vertex is a meeting of two octagons and one triangle. This solid should appear like a cube of which the corners are sliced away to leave equilateral triangles in their place.

The thirteen Archimedian solids differ by the kind and number of polygons which meet at their vertices:

Name	n-gons at each vertex	number of sides
Snub Cube	4 tri and 1 quad	38 sides
Cubooctahedron	2 tri and 2 quad	14 sides
Rhombicuboctahedron	1 tri and 3 quad	24 sides
Snub decahedron	4 tri and 1 penta	92 sides
Icosidodecahedron	2 tri and 2 penta	32 sides
Truncated tetrahedron	1 tri and 2 hex	8 sides
Truncated cube	2 oct and 1 tri	14 sides
Truncated dodecahedron	2 dec and 1 tri	22 sides
Truncated octahedron	1 squ and 2 hex	14 sides
Truncated icosahedron	1 penta and 2 hex	22 sides
Trun.isosidodecahedral rhombus	1 tri, 2 quad and 1 penta	62 sides
Icosihexahedron	1 quad,1 hex and 1 oct	26 sides
Trun.icosadodecahedron	1 quad,1 hex and 1 deca	62 sides

4.4 Rhombic solids

A rhombic solid is a polyhedron having faces which are rhombi all congruent to one another, although in a manner slightly deficient to that of the congruence of regular polygons. Excluding the cube as a rhombic solid, two rhombic solids are claimed to exist. One should have twelve faces, and the other should have thirty. The rhombi of the two solids should differ in the ratios of their diagonals. Although on either individual solid the faces have all the same ratio and so are congruent, at any vertex of either solid, the angles cannot be all congruent.

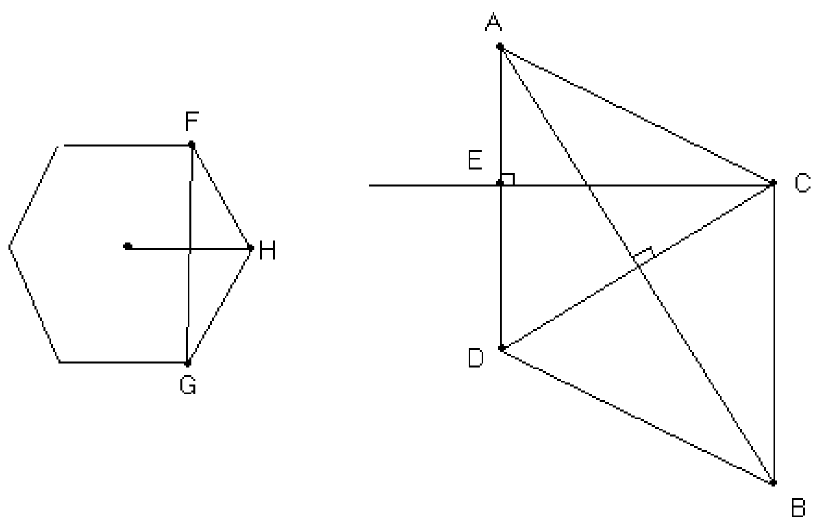
4.5 Construction problems of the rhombic solids.

1. Suppose that twelve congruent rhombi could be assembled into a single solid in exactly one way. Would the solid angles be all congruent? For example, would every solid angle be a meeting of two obtuse angles and one acute angle?

2. Suppose that one rhombic solid could be described as follows:

Six faces form a hexagonal column. The rhombi are everywhere oriented so that acute angles meet acute angles and obtuse meet obtuse. Three more faces close an end of the column and an acute angle from each meets to form a single vertex. Three more faces close the other end of the column in a similar way.

Using a trigonometric argument, determine the ratio of the two diagonals of a rhombus which would be used in this construction. Alternatively, if the ratio is not unique, describe the set of ratios that would permet such a construction.



Claim: Given six rhombi having diagonals of ratio $1 : \sqrt{2}$ and being arranged in a hexagonal "crown", the crown may be closed by similar rhombi; i.e., show that the closing rhombi have diagonals in the same ratio.

Sketch of proof: Let ACBD be a rhombus such that $|\mathsf{AB}| = \sqrt{2}$ and $|\mathsf{CD}| = 1$. Let $\mathsf{EC} \perp \mathsf{AD}$. Then $|\mathsf{AC}| = \sqrt{\frac{2}{4} + \frac{1}{4}} = \frac{1}{2}\sqrt{3}$. Then $\angle \mathsf{ACD} = \cos^{-1}(\frac{1}{2})\frac{1}{|\mathsf{AC}|}$. So, $\angle \mathsf{BAC} = 90 - \angle \mathsf{ACD}$. Thus, $\angle \mathsf{EAB} = 2\angle \mathsf{BAC}$. Then

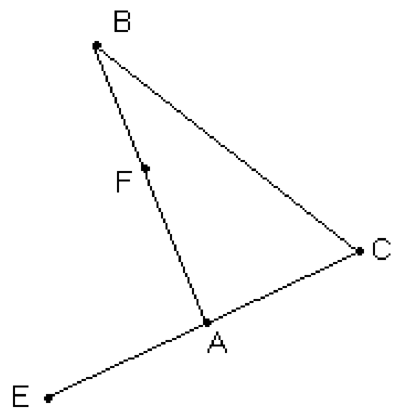
$|EC| = (\cos\angle ACE)|AC| = \sqrt{\frac{2}{3}}$ Then $|FG| = 2\sqrt{|EC|^2 - \frac{1}{4}|EC|^2} = |EC|\sqrt{3} = \sqrt{2}$. Thus the top rhombi have a diagonal of length $\sqrt{2}$. Supposing that they share sides with the six fixed rhombi, the other diagonal would have to be of length one. QED

5 The divine section

5.1 Constructing the divine section

1. Let segment AB be the one to be divided
2. Construct segment BC normal to AB so that $|\mathsf{BC}| = \frac{|\mathsf{AB}|}{2}$
3. Construct segment CE along CA and so that $|\mathsf{CE}| = |\mathsf{BC}|$
4. Construct segment AF along AB and so that $|\mathsf{AF}| = |\mathsf{AE}|$

Then AD:AB::BD:AD by a simple argument. Thus D divides AB into the "divine" section (half the difference between the square root of five and one is equal to both ratios, the smaller to the larger and the larger to the whole).



Claim: $\frac{|BF|}{|FA|} = \frac{|FA|}{|BA|} = \frac{\sqrt{5}-1}{2}$

Proof:

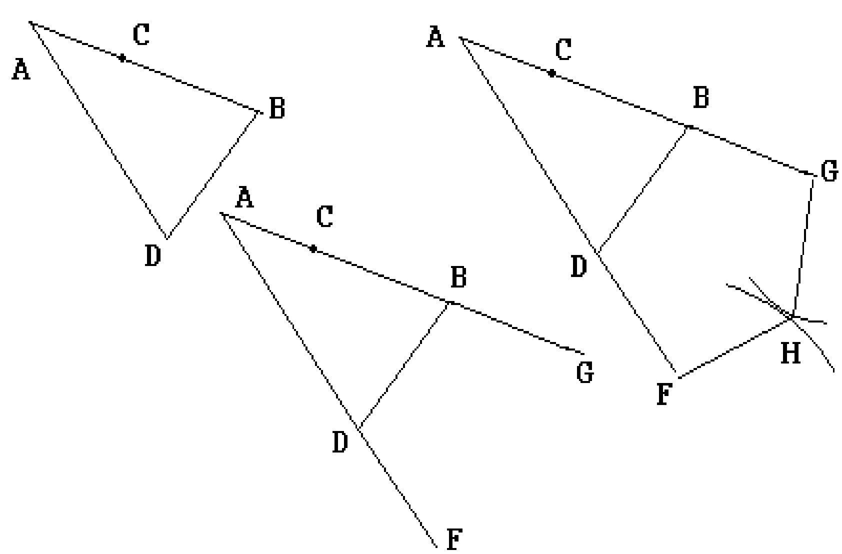
$|FA| = |BC| - |AC|$. Then, $|BC| - |AC| = \sqrt{|AB|^2 + |AC|^2} - |AC|$ by pythagorean theorem. Since $|AB| = 2|AC|$, $|FA| = |AB|\sqrt{1 + \frac{1}{4}} - \frac{1}{2}|AB| = |AB|(\frac{\sqrt{5}}{2} - \frac{1}{2})$, so $\frac{|FA|}{|AB|} = \frac{\sqrt{5}-1}{2}$. Since this is the divine section, immediately the result follows that $\frac{|BF|}{|FA|} = \frac{|FA|}{|BA|}$. QED

5.2 The regular pentagon

The regular pentagon is one of the constructible polygons, and its construction involves the use of the divine section.

To construct a regular pentagon:

Let AB be a segment divided into the divine section at point C. Construct an isosceles triangle $\triangle \mathsf{ABD}$ so that $|\mathsf{AB}| = |\mathsf{AD}|$ and $|\mathsf{CB}| = |\mathsf{BD}|$. Construct segments BF and DG as extentions of AB and AD, respectively so having the length of CB. Construct point H as the intersection of circles at G and F having radius CB. Segments GH and FH then complete the pentagon.



Claim: BGHFD is a regular pentagon.

Proof: Clearly from the construction, the sides of the pentagon are all the same length, so what suffices is to show that the interior angles of the figure are all congruent and that they are each $108°$, which is the measure of the interior angles of the regular pentagon.

$\cos\angle \mathsf{ABD} = \frac{\frac{1}{2}|\mathsf{BD}|}{|\mathsf{AB}|} = \frac{1}{2}(\frac{\sqrt{5}-1}{2})$. So, $\angle \mathsf{ABD} = \cos^{-1}(\frac{\sqrt{5}-1}{4}) = 72°$. Since $\angle \mathsf{GBD} = 180° - \angle \mathsf{ABD}$, then $\angle \mathsf{GBD} = 108°$. The claim follows by symmetry.

QED

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